Response of First Order Systems to Sinusoidal Input

First Order System - Sinusoidal Input

Consider the following first order system:

$$G(s) = \frac{O(s)}{I(s)} = \frac{1}{\tau s + 1}$$

Determine the response of the system if input is a sinusoidal:

$$I(t) = A\sin(\omega t)$$

Which may be transformed to:

$$I(s) = \frac{A\omega}{s^2 + \omega^2}$$

the system response, *O*(s), is then:

$$O(s) = \frac{A\frac{\omega}{\tau}}{(s+\frac{1}{\tau})(s^2+\omega^2)} = \frac{a}{s+\frac{1}{\tau}} + \frac{b}{s+i\omega} + \frac{c}{s-i\omega}$$

Solving then for a,b, and c:

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To solve for a, multiply by s + 1/τ , and let s = -1/τ

$$\frac{A\frac{\omega}{\tau}}{\frac{1}{\tau^2}+\omega^2} = a = \frac{A\tau\omega}{1+\tau^2\omega^2}$$

To solve for b, multiply by $s + i\omega$, and let $s = -i\omega$

$$\frac{A\frac{\omega}{\tau}}{(-i\omega+\frac{1}{\tau})(-i\omega-i\omega)} = \frac{A\frac{\omega}{\tau}}{2i^2\omega^2-\frac{2i\omega}{\tau}} = \frac{A}{i^2 2\tau\omega-2i} =$$

$$b = \frac{-\frac{A}{2}}{\tau\omega+i}\left(\frac{\tau\omega-i}{\tau\omega-i}\right) = \frac{-\frac{A}{2}(\tau\omega-i)}{\tau^2\omega^2+1}$$

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c will be the complex conjugate of b and is:

$$c = \frac{-\frac{A}{2}(\tau\omega + i)}{\tau^2\omega^2 + 1}$$

Using the solution in the S&C for a sinusoidal function, the solution becomes:

$$L^{-1}\left[\frac{B+iC}{s-r-i\omega}\frac{B-iC}{s-r+i\omega}\right] = 2\sqrt{B^2+C^2}e^{rt}\sin(\omega t+\phi)$$

Where: $\phi = \tan^{-1}\left(\frac{C}{B}\right)$ $C = \left(\frac{A}{\tau^2\omega^2+1}\right)\left(\frac{1}{2}\right)$ $B = \left(\frac{-A}{\tau^2\omega^2+1}\right)\left(\frac{\tau\omega}{2}\right), r = 0$

$$2\left(\frac{A}{\tau^2\omega^2+1}\right)\left(\frac{1}{2}\right)\sqrt{\tau^2\omega^2+1}\left(e^{0t}\right)\sin(\omega t+\phi) \qquad \phi = \tan^{-1}(-\tau\omega)$$

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The complete solution is then:

$$O(t) = \frac{A\tau\omega}{\tau^2\omega^2+1}e^{\frac{t}{\tau}} + \left(\frac{A}{\sqrt{\tau^2\omega^2+1}}\right)\sin(\omega t + \phi)$$

It is important to note that the solution is made-up of a transient (the first term) and a non-transient part (the second term).

Consider simply substuting $\imath\omega$ for s in the original transfer function and solving for $|G(i\omega)|$ $\angle G(i\omega)$

$$G(i\omega) = \frac{1}{\tau i\omega + 1} \quad G(i\omega) = \frac{1}{\tau i\omega + 1} = \frac{i\tau\omega - 1}{i^2\tau^2\omega^2 - 1} = -\frac{i\tau\omega - 1}{\tau^2\omega^2 + 1} = \frac{1 - i\tau\omega}{\tau^2\omega^2 + 1}$$

$$|G(i\omega)| = \sqrt{\mathrm{Im}(G(i\omega)) + \mathrm{Re}(G(i\omega))} = \sqrt{\left(\frac{\tau\omega}{\tau^2\omega^2+1}\right)^2 + \left(\frac{1}{\tau^2\omega^2+1}\right)^2} = \frac{1}{\sqrt{\tau^2\omega^2+1}}$$

$$\angle G(i\omega) = \tan^{-1}\left(\frac{\mathrm{Im}(G(i\omega))}{\mathrm{Re}(G(i\omega))}\right) = \tan^{-1}(-\tau\omega)$$

Note, this gives us the non-transient solution

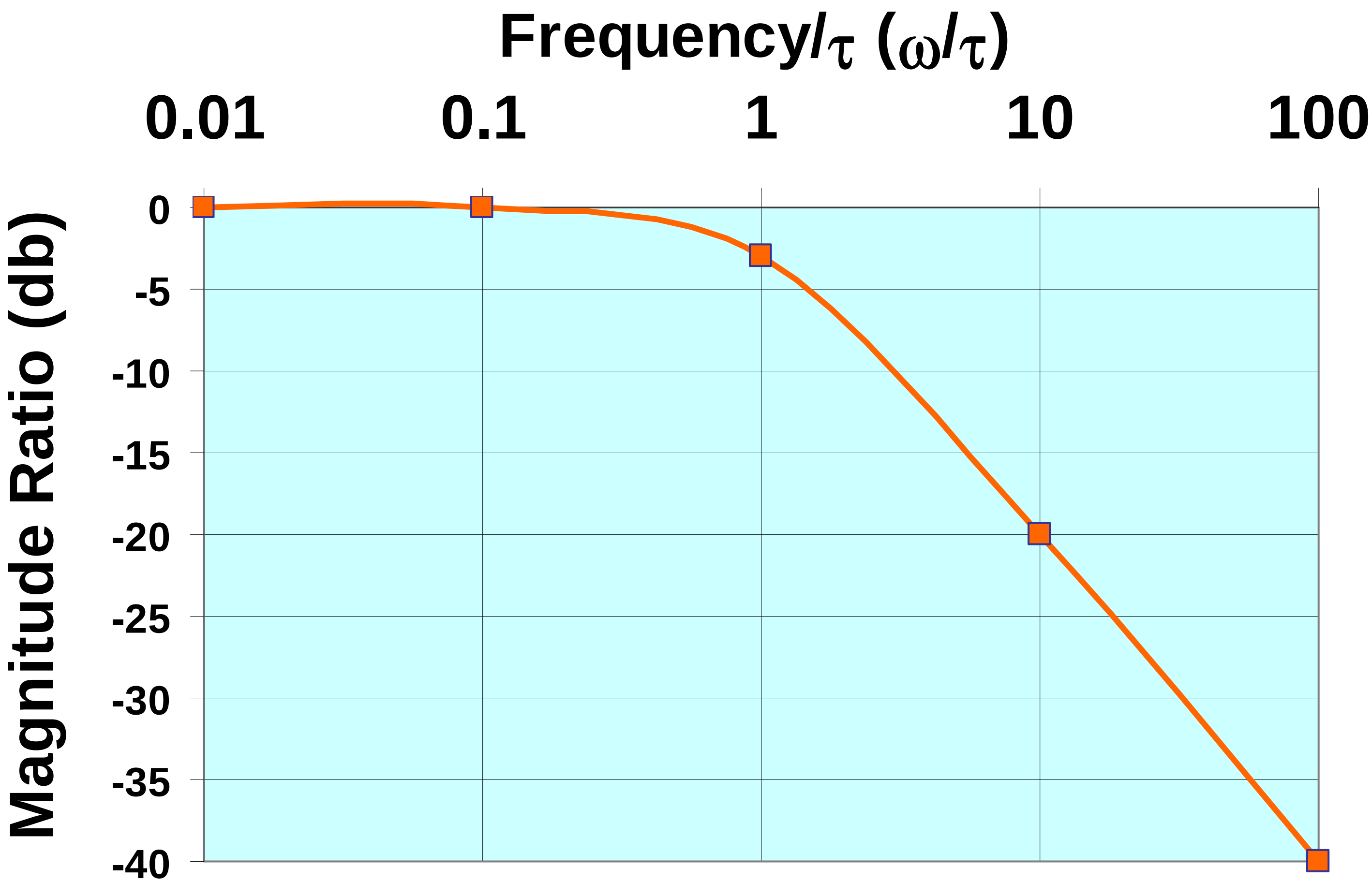
Plotting Frequency response

$$\left|G(i\omega)\right| = \frac{1}{\sqrt{\tau^2\omega^2+1}}$$

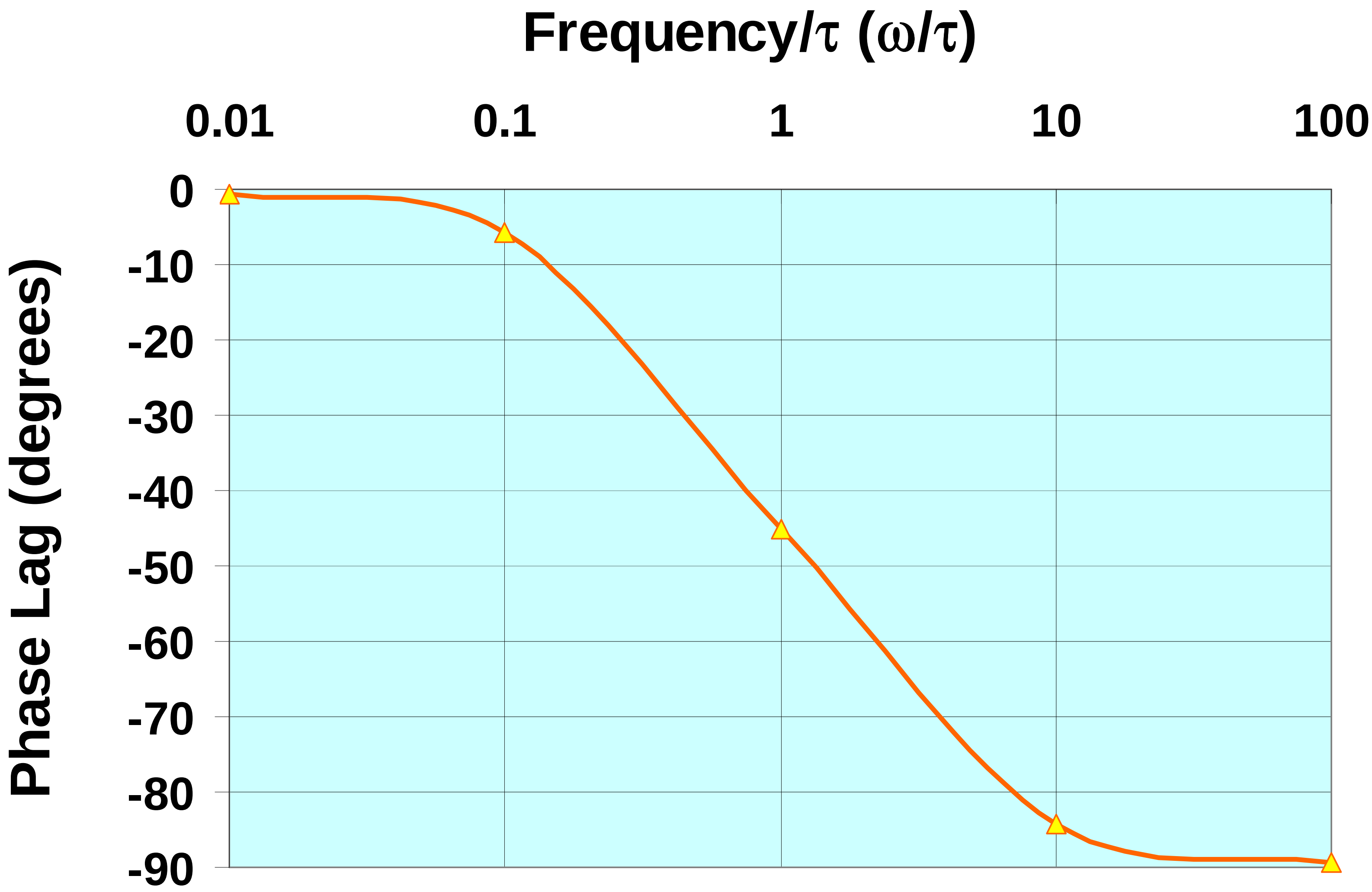
$$\angle G(i\omega) = \tan^{-1}\left(-\tau\omega\right)$$

ω	**G(iω)**	**20log(G(iω))**	φ
$0.01/\tau$	**1**	**0**	-0.5729
$0.1/\tau$	**1**	**0**	-5.7106
$1/\tau$	**0.7071**	**-3**	-45
$10/\tau$	**0.1**	**-20**	-84.289
$100/\tau$	**0.01**	**-40**	-89.427

Plotting Frequency response



Plotting Frequency response



Adding terms of Frequency Response

$$20\log\left(\left|G_1(i\omega)\right|\left|G_2(i\omega)\right|\right) = 20\log\left(\left|G_1(i\omega)\right|\right) + 20\log\left(\left|G_2(i\omega)\right|\right)$$

$$\angle G_1(i\omega)G_2(i\omega) = \angle G_1(i\omega) + \angle G_2(i\omega)$$

We can simply add terms on the Bode Magnitude plot and on the Bode Phase plot to get total response