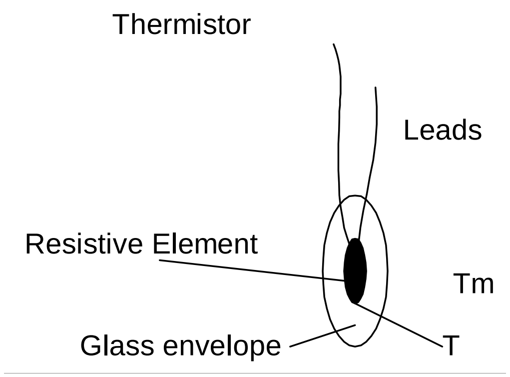
First Order Response Characteristics

Modeling Thermistor Response

Heat Balance

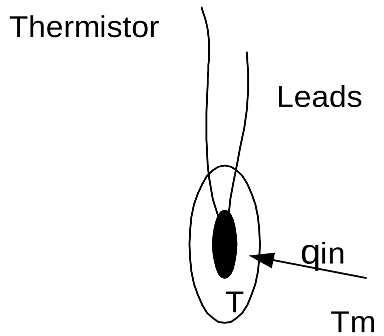
Given, a thermistor device with a glass envelope surrounding a small resistive element as shown below. The glass envelope and resistive element is assumed to have a mass M and a specific heat C_p. Calculate the temperature of the thermistor, T, given the environmental temperature Tm is stepped by 10 ^oC.



The first law of thermodynamics (conservation of heat and work) can be applied to solve this problem and will be written in the form: Inflow rate - Outflow rate = Accumulation rate +Work rate. For this problem no work is being done:

$q_{in} - q_{out} = \frac{dU}{dt}$ Where q is heat flow rate and U is internal energy.

Heat Transfer



The heat flow into the thermistor is assumed positive entering the thermistor. Paths through which heat is lost from the thermistor might be through radiation or through the leads. Both are assumed to be zero for this preliminary analysis. Heat flow into the thermistor can be calculated using general conduction/convection heat transfer model with a universal heat transfer coefficient:

$$q_{in} = U_h A(T_m - T)$$

where U_h is the universal heat transfer coeficient, and A is the effective surface area of the thermistor.

The change in internal energy of the thermistor can be calculated as: $dU = MC_p dT$ where no phase changes are apparent. The equation describing the system then becomes:

$$U_h A(T_m - T) = MC_p \frac{dT}{dt}$$

Deviation Variables

Rearranging, $T_m = \frac{MC_p}{U_h A}\frac{dT}{dt} + T$

The equation can be written in terms of deviation variables to assure later that initial conditions will be zero. In order to do that the following assumptions will be used:

$$\theta = T - T_{ss}$$
$$d\theta = d(T - T_{ss}) = dT$$

where the ss indicates the steady state or initial value.

Writing the steady state equation from : $T_m = \frac{MC_p}{U_h A}\frac{dT}{dt} + T$, we have: $T_{mss} = \frac{MC_p}{U_h A}\frac{dT_{ss}}{dt} + T_{ss}$,

or: $T_{mss} = T_{ss}$

subtracting this steady state equation from the original differential equation and letting $dT = d\theta$, we have:

$$\theta_m = \frac{MC_p}{U_h A}\frac{d\theta}{dt} + \theta$$

Partial Fractions

This equation can be readily solved for θ where $\theta_m = 10u(t)$ (a step in input temperature). Note that θ_m is a deviation variable and that $10u(t)$ represents the change in temperature we originally wanted as an input. Taking the Laplace Transform of the equations, we have:

$$\theta_m(s) = \frac{MC_p}{U_h A} s\theta(s) + \theta(s) \qquad \theta_m(s) = \frac{10}{s}$$

Simplifying we have the transfer function relating output to input:

$$\frac{\theta(s)}{\theta_m(s)} = \frac{1}{\left[\frac{MC_p}{U_h A} s + 1\right]}$$

Solving for θ we have:

$$\theta(s) = \frac{10}{s\left[\frac{MC_p}{U_h A} s + 1\right]}$$

Using a partial fraction expansion, we can express the above equation as:

$$\theta(s) = \frac{10}{s\left[\frac{MC_p}{U_h A} s + 1\right]} = \frac{a}{s} + \frac{b}{\left[\frac{MC_p}{U_h A} s + 1\right]}$$

Inverse Transformation

Solving for a, and b by multiplying through by the denominator of each factor and then letting s equal the value that forces that denominator to become zero, a, and be are:

$$\frac{10}{1} = a \text{ , and } \frac{10}{-\frac{U_h A}{MC_p}} = b = -10\frac{MC_p}{U_h A}$$

Theta then becomes:

$$\theta(s) = \frac{10}{s} - \frac{10\frac{MC_p}{U_h A}}{\left[\frac{MC_p}{U_h A}s + 1\right]} = \frac{10}{s} - \frac{10}{\left[s + \frac{U_h A}{MC_p}\right]}$$

Transforming back to the time domain, then $\theta(t)$ then becomes:

$$\theta(t) = 10u(t) - 10u(t)e^{-\frac{U_h A}{MC_p}t} = 10u(t)\left[1 - e^{-\frac{U_h A}{MC_p}t}\right]$$

Solution

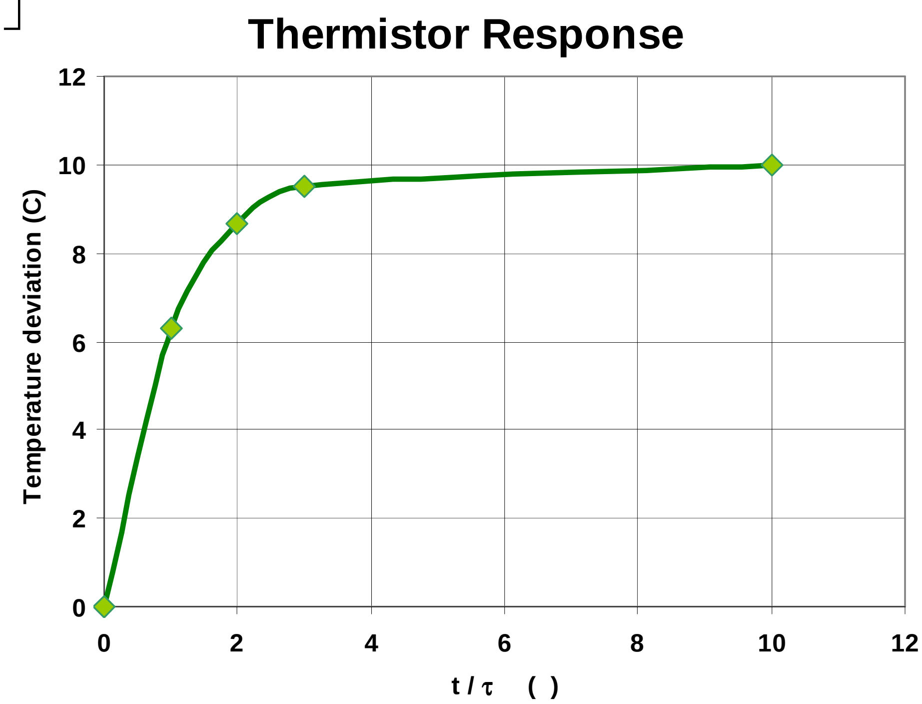
Transforming back to the time domain, then $\theta(t)$ then becomes:

$$\theta(t) = 10u(t) - 10u(t)e^{-\frac{U_h A}{MC_p}t} = 10u(t)\left[1 - e^{-\frac{U_h A}{MC_p}t}\right]$$

Letting: $\dfrac{MC_p}{U_h A} = \tau$, $\theta(t) = 10u(t)\left[1 - e^{-\frac{t}{\tau}}\right]$

Theta may be plotted with respect to time.

t	t/τ	$1-e^{-t/\tau}$	$10*(1-e^{-t/\tau})$
0	0	0.00	0.00
τ	1	0.63	6.32
2τ	2	0.86	8.65
3τ	3	0.95	9.50
10τ	10	1.00	10.00



First order systems

A standard form for the transfer function of a first order system is:

$$G(s) = \frac{K}{\tau s + 1} = \frac{Output(s)}{Input(s)}$$

Where:

K is the “Steady State Gain” of the system and it describes the output per unit of input when the system comes to steady state.

τ is known as the “time constant” and it describes the time required for a first order system to complete 63.2% of its eventual response to a step input.

This relationship may also be expressed as a block diagram as follows: